

SUPERSYMMETRIC BLACK HOLES IN $N = 8$ SUPERGRAVITY

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Abstract

We study the embedding of extreme (multi-) dilaton black hole solutions for the values of the parameter $a = \sqrt{3}, 1, 1/\sqrt{3}, 0$ in $N = 4$ and $N = 8$ four-dimensional supergravity. For each black hole solution we find different embeddings in $N = 4$ supergravity which have different numbers of unbroken supersymmetries. When embedded in $N = 8$ supergravity, all different embeddings of the same solution have the same number of unbroken supersymmetries. Thus, there is a relation between the value of the parameter a and the number of unbroken supersymmetries in $N = 8$ supergravity, but not in $N = 4$, and the different embeddings must be related by dualities of the $N = 8$ theory which are not dualities of the $N = 4$ theory. The only exception in this scheme is a *dyonic* embedding of the $a = 0$ black-hole solution which seems to break all supersymmetries both in the $N = 4$ and in the $N = 8$ theories.

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1 Introduction

The abundance of extreme black-hole and soliton solutions of string theory is intimately connected with the existence of duality symmetries either within a given string theory or between different string and higher-membrane theories (see [1, 2, 3, 4] and references therein). These duality symmetries can always be used as solution-generating transformations which generate new solutions out of known solutions, and, therefore, the larger the duality group, the greater the number of extreme black-hole and soliton solutions.

There is, however, another point of view, which may be regarded as “dual” to this one: solutions related by duality are in some sense equivalent (some times they are equivalent in a strict sense), so that solutions generated by duality transformations are not regarded as completely new. From this latter viewpoint, one is particularly interested in solutions which are not related by any duality symmetry since new solutions can then be generated from this reduced set of inequivalent solutions. In this scenario, the larger the group of duality symmetries, the smaller the set of inequivalent solutions.

This is the point of view that we are going to adopt in this paper. We consider extreme black-hole solutions arising in four-dimensional $N = 4$ and $N = 8$ supergravity reductions of $N = 1$ and $N = 2$ ten-dimensional supergravities. Our main goal is the following: we want to find how many inequivalent extreme dilaton black-hole solutions there are in these theories, with just one scalar and one vector field in four dimensions and coupling between the scalar and the vector field characterized by the constant a . Equivalent solutions should be related by duality symmetries and should have the same number of unbroken supersymmetries.

In fact, the interplay between duality and supersymmetry, both spacetime and worldsheet, has been the subject of active investigation recently. In particular, it has been noted that soliton solutions of a given theory which transform into each other under T duality preserve the same amount of supersymmetry under most circumstances [5]. $N = 4, D = 4$ Killing spinors transform covariantly under S duality, the number of unbroken supersymmetries and Bogomol’nyi bounds being invariant [6, 7], and similar results have been obtained in the context of $N = 4, D = 8$ supergravity [8]. More recently, an analysis of the supersymmetry-breaking pattern of string-like solitons in toroidally compactified four-dimensional heterotic string theory [9] showed that such solitons which are related by an $O(8, 24; \mathbb{Z})$ transformation (this larger duality group containing both the $SL(2, \mathbb{Z})$ S duality group and the $O(6, 22; \mathbb{Z})$ T duality group) preserved the same amount of spacetime supersymmetry, whether it was $1/2$, $1/4$ or $1/8$ of the original supersymmetries of the theory.

Thus, supersymmetry can be used to classify inequivalent extreme black-hole solutions.

Most of the extreme black-hole solutions discussed so far have been found to preserve precisely half of the supersymmetries of whatever low-energy theory they arise in: extreme Reissner-Nordström black holes break one half of the $N = 2, D = 4$ supersymmetries (see, for instance, the second lecture of Ref. [10] and references therein), ($a = 1$) extreme dilaton black holes break one half of the $N = 4, D = 4$ supersymmetries [11], Kaluza-Klein black holes break one half of the supersymmetries of $N = 1, D = 5$ supergravity since they correspond to five-dimensional pp -waves (see, for instance, the third lecture of Ref. [10] and references therein). Nevertheless, there exist several examples of solutions which break $3/4$ of the supersymmetries or more, such as the double-instanton string of [12] and most of the string-like solitons of [9] (see also [2] and references therein for other examples).

However, the statement that the ($a = 1$) extreme dilaton black hole has one half of the $N = 4, D = 4$ supersymmetries unbroken (that is half of $N = 1, D = 10$) is not, in fact, true, or, more precisely, is incomplete. One should say that there is an *embedding* of the ($a = 1$) extreme dilaton black hole which has one half of the $N = 4, D = 4$ supersymmetries unbroken. This is the embedding proposed in Ref. [11] and basically corresponds to the identification of the vector field with a vector field belonging to the gravity supermultiplet of $N = 4, D = 4$ supergravity. There is, however, another embedding of the same solution which is *not* supersymmetric, and which corresponds to the identification of the vector field with one belonging to an $N = 4, D = 4$ vector supermultiplet³ [13].

These two embeddings are, according to the above results on duality and supersymmetry, inequivalent in the framework of the $N = 4$ theory, in spite of the fact that they correspond to identical solutions satisfying identical Bogomol’nyi-like bounds.

This situation seems to us a bit paradoxical. One would naively think that all the possible embeddings of a given solution should be equivalent. One would also naively think that all Bogomol’nyi-like bounds should be related to supersymmetry, at least in some theory.

As we will see, the resolution of this apparent paradox lies in the $N = 8$ theory. All embeddings in the $N = 4$ theory can also be considered as embeddings in the $N = 8$ theory, and we will only consider this kind of simultaneous (or NS-NS) embedding. With the exception of a new dyonic black hole which breaks all of the supersymmetries [14], our findings indicate that there is a well-defined relation between the parameter a and the number of supersymmetries preserved in $N = 8$ supergravity. For either purely electric or purely magnetic four-dimensional configurations, all NS-NS embeddings of the same extreme black-hole solution have the same number of unbroken supersymmetries and this suggests that all

³This vector supermultiplet can have a Kaluza-Klein origin, appearing in the dimensional reduction of $N = 1, D = 10$ supergravity or can be related to one of the sixteen ten-dimensional $U(1)$ vector supermultiplets of the heterotic string.

embeddings are equivalent in the framework of the $N = 8$ theory. Some of the $N = 8$ dualities (U dualities [3]) that connect the different embeddings are not present in the $N = 4$ theory, and so it follows that these embeddings are not ($N = 4$)-equivalent.

Particularly interesting is the U duality transformation that we call C duality whose effect is to interchange supergravity and matter vector fields and the chirality of the ten-dimensional spinors [15]. The embeddings proposed in Refs. [13, 11] are related by C duality and therefore are both supersymmetric in the $N = 8$ theory. However, $N = 1, D = 10$ supergravity is chiral, and C duality cannot be a duality symmetry of the $N = 4$ theory. In fact, C duality is a string/string duality relating the two $N = 1$ supergravity theories with opposite chiralities that one can construct in ten dimensions. This explains the apparent paradox.

Four-dimensional Kaluza-Klein black holes have been studied in Refs. [16], while four-dimensional black-hole solutions of $N = 8$ supergravity have also been studied in Refs. [17]. Furthermore, in Refs. [18], a class of regular BPS saturated black holes parameterized by four charge parameters was constructed, while in Refs. [19] a five-parameter construction of all static, spherically symmetric BPS-saturated black holes of heterotic string theory compactified on a six-torus was shown. While our solutions appear to be special examples of the abovementioned generalized constructions, our work differs from these in that we are interested in the interplay between U duality and the different truncations of $N = 8$ supergravity. The appearance of supersymmetric extreme solutions which saturate a Bogomol’nyi bound and non-supersymmetric extreme solutions which do not saturate a Bogomol’nyi bound in the context of the four-parameter solution of Refs. [18] was shown in Refs. [20] to arise from the different limits from non-extremality of the four-parameter solution. This observation does not, however, resolve the apparent paradox in question in this paper, as all the classes of black hole solutions we consider do in fact saturate a Bogomol’nyi-like bound.

A summary of our work is as follows: in Section 2 we write down the action and multi-black hole solutions of the a -model, noting for which values of a the solutions arise from consistent truncations of maximal $N = 8$ supergravity. In Section 3 we demonstrate the embeddings, both known and novel, of the a -model solutions in $N = 4$ and $N = 8$ supergravity, and proceed in Section 4 to find the unbroken supersymmetries for each embedding. Finally, we discuss our results in Section 5.

The appendices contain some complementary material and results that we heavily use in the main body of the paper: our conventions are in Appendix A, in Appendix B we derive from eleven dimensions the ten-dimensional supersymmetry rules of the type IIA theory and of the two $N = 1$ theories of opposite chiralities, together with the C duality relation between them. Appendix C contains the spin connection one-form for the ten-dimensional class of metrics which

we consider.

2 The “ a -model”

In reducing to four dimensions the ten-dimensional $N = 1$ and $N = 2$ supergravities arising in the low-energy limit of the various superstring theories, one generically arrives at a complicated four-dimensional action with many scalar fields (moduli) and Maxwell vector fields (throughout this work we consider Abelian vector fields only, as even when we start with a non-Abelian gauge group, a generic point in the moduli space has $U(1)^n$ symmetry). For example, toroidally compactified four-dimensional heterotic string theory consists of a metric, antisymmetric tensor, dilaton, 28 vectors and 132 scalars.

In this paper we consider a greatly simplified truncation in four-dimensions, consisting of a metric $\tilde{g}_{\mu\nu}$, a single scalar field φ and a single Maxwell field A_μ with an arbitrary parameter a governing the scalar-Maxwell coupling. The “ a -model” action is explicitly given by

$$S^{(a)} = \frac{1}{2} \int d^4x \sqrt{|\tilde{g}|} \left[-\tilde{R} - 2(\partial\varphi)^2 + \frac{1}{2}e^{-2a\varphi}F^2 \right]. \quad (1)$$

We stress that the scalar φ is in general different from the string dilaton, and is in fact a linear combination of the dilaton and other moduli. Throughout we always denote the dilaton by a different symbol (ϕ). Therefore, in working with this model, we are always in the canonical (Einstein) frame, which we denote with tildes on the metric, Einstein tensor etc. The equations of motion are

$$\left. \begin{aligned} \tilde{G}_{\alpha\beta} + 2T_{\alpha\beta}^\varphi - e^{-2a\varphi}T_{\alpha\beta}^A &= 0, \\ \tilde{\nabla}^2\varphi - \frac{a}{4}e^{-2a\varphi}F^2 &= 0, \\ \tilde{\nabla}_\mu(e^{-2a\varphi}F^{\mu\alpha}) &= 0, \end{aligned} \right\} \quad (2)$$

where $T_{\alpha\beta}^\varphi$ and $T_{\alpha\beta}^A$ are the energy-momentum tensors of the scalar field φ and the vector field A_μ respectively:

$$T_{\alpha\beta}^\varphi = \partial_\alpha\varphi\partial_\beta\varphi - \frac{1}{2}\tilde{g}_{\alpha\beta}(\partial\varphi)^2, \quad (3)$$

$$T_{\alpha\beta}^A = F_\alpha{}^\mu F_{\beta\mu} - \frac{1}{4}\tilde{g}_{\alpha\beta}F^2. \quad (4)$$

Black-hole solutions of the a -model exist for all values of the parameter a , that we take to be positive without any loss of generality. In particular, there

are extreme [21] and multi-black-hole solutions [22, 6] for all a . The purely electric extreme multi-black-hole solutions are

$$\begin{cases} d\tilde{s}^2 &= V^{-\frac{2}{1+a^2}} dt^2 - V^{+\frac{2}{1+a^2}} d\vec{x}^2, \\ e^\varphi &= V^{-\frac{a}{1+a^2}}, \\ F_{t\bar{i}} &= -n \sqrt{\frac{2}{1+a^2}} \partial_{\bar{i}} V^{-1}, \end{cases} \quad (5)$$

where $V(\vec{x})$ is a harmonic function in three-dimensional Euclidean space

$$\partial_{\bar{i}} \partial_{\bar{i}} V = 0, \quad (6)$$

that we always take to be positive and normalized so as to make the above metric regular and asymptotically flat, that is

$$V(\vec{x}) = 1 + \sum_i \frac{M_i}{|\vec{x} - \vec{x}_i|}, \quad M_i \geq 0 \quad \forall i, \quad (7)$$

and $n = \pm 1$ gives the sign of the electric charges. The equations of motion of the a -model are invariant under the discrete electric-magnetic duality transformation

$$F' = e^{-2a\varphi} F, \quad \varphi' = -\varphi, \quad (8)$$

and, therefore, a purely magnetic multi-black-hole solution always exists for any a :

$$\begin{cases} \tilde{d}s^2 &= W^{-\frac{2}{1+a^2}} dt^2 - W^{+\frac{2}{1+a^2}} d\vec{x}^2, \\ e^\varphi &= W^{+\frac{a}{1+a^2}}, \\ F_{\bar{i}\bar{j}} &= \mp \sqrt{\frac{2}{1+a^2}} \epsilon_{ijk} \partial_{\bar{k}} W. \end{cases} \quad (9)$$

For the special values $a = 0$ and $a = 1$ dyonic solutions also exist [23, 11]:

$$\begin{cases} \tilde{d}s^2 &= (VW)^{-1} dt^2 - VW d\vec{x}^2, \\ e^\varphi &= V^{-\frac{1}{2}} W^{+\frac{1}{2}}, \\ F &= n dV^{-1} \wedge dt - \frac{1}{2} m \epsilon_{ijk} \partial_{\bar{i}} W dx^{\bar{j}} \wedge dx^{\bar{k}}, \end{cases} \quad (10)$$

where n and m take the values ± 1 . (The $a = 0$ dyon is obtained by setting $V = W$ in the above solution.)

All the purely electric or magnetic extreme solutions (and the dyonic $a = 1$, $a = 0$ solutions) admit Killing spinors if one uses the appropriate definition of “gravitino” and “dilatino” supersymmetry transformation rules [24]. These rules coincide with the supersymmetry rules of known supergravity theories in some cases, and they can always be used to do Nester constructions. It is worth noting, though, that all the supersymmetry rules of these subsupergravities can be obtained from the $N = 4$, $D = 4$ supersymmetry rules (with no axion) through the same transformation that takes the $a = 1$ extreme dilaton black hole into the other values of a (see the conclusions section of Ref. [6]).

However, it is not known which values of a naturally appear in *true* supergravity theories. As explained in Ref. [3], some of them are expected to arise in the different consistent truncations of maximal $N = 8$ supergravity, namely those with $a = \sqrt{3}, 1, 1/\sqrt{3}, 0$. The values 1 and 0 arise in the truncations to $N = 4$ and $N = 2$ respectively. The values $\sqrt{3}$ [25] and $1/\sqrt{3}$ arise in the truncation from maximal to simple five-dimensional supergravity and its dimensional reduction to $D = 4$. For all four values of a one can also argue that the extreme black-hole solutions are solitonic.

A study of the slow motion of the extreme black-hole solutions of the a -model reveals that only for $a = \sqrt{3}$ the corresponding moduli space is flat [26, 27]. This is the necessary condition for the solutions to only break half of the $N = 8$ supersymmetries [28] (*i.e.* to be BPS states), and therefore we only expect these to have half of the $N = 8$ supersymmetries unbroken and the rest will have fewer unbroken supersymmetries.

It was conjectured in [29] that certain electrically charged extreme black holes arising in the $N = 4$ theory could be identified with BPS states in the spectrum of allowed charges of the theory [30] (the so-called Schwarz-Sen spectrum), which in turn could be identified with elementary (massive) string states. In the $N = 4$ theory, both the $a = \sqrt{3}$ black hole and a certain embedding of the $a = 1$ black hole preserve half of the spacetime supersymmetries, and were shown to correspond to massive string states (dynamical evidence supporting the conjecture in [29] was found in [31]).

In the $N = 8$ theory, however, only $a = \sqrt{3}$ black holes preserve half the supersymmetries, and are therefore the only candidates to be identified with elementary string states. On the other hand, all four black holes (at least in some embeddings) preserve some degree of supersymmetry, and saturate Bogomol'nyi bound formulae (see, for example, [18, 32]). In truncating to an $N = 2$ theory, one can find embeddings for all four black holes such that each preserves half of the spacetime supersymmetries. As a result, there is the possibility of realizing all these extremal black holes as string states, although in the $N = 2$ case the solutions are no longer protected by nonrenormalization theorems against

quantum corrections.

The problem with quantum corrections arises especially for the $a = 0$ black hole, since this solution has zero dilaton for both electric and magnetic solutions, and there is no way to distinguish a perturbative from a non-perturbative state. One is then led to conclude that both solutions are non-perturbative, and cannot correspond to an elementary string state to begin with⁴.

On the other hand, it was also conjectured in Ref. [29] that the $a = 1, 1/\sqrt{3}, 0$ extreme dilaton black holes could be understood as bound states of the 2, 3 and 4 maximally supersymmetric $a = \sqrt{3}$ black holes respectively. This conjecture has been recently confirmed in Ref. [33] where it was shown how to get the $a = 1, 1/\sqrt{3}, 0$ solutions as particular limits of a multi- $a = \sqrt{3}$ -black-hole solution that interpolates between them. In our supersymmetry analysis, a similar picture of compositeness arises in relating the Killing spinor equations of the various black holes. For example, as we shall see below, the supersymmetry breaking pattern of the $a = 1/\sqrt{3}$ black hole, corresponding to 3 $a = \sqrt{3}$ black holes in the bound state picture, arises as the combination of the supersymmetry breaking of a single $a = \sqrt{3}$ black hole and an $a = 1$ black hole, corresponding to 2 $a = \sqrt{3}$ black holes in the bound state picture.

Additional information on the a -model comes from its reduction to two dimensions. For (and only for) the special values $a = 0, \sqrt{3}$ the two dimensional theory has infinite symmetry and becomes completely integrable [34].

Our purpose in the next section is to investigate which solutions of the a -model do arise in $N = 4(8)$, and for which values of a , how they are embedded in the supergravity theory and their unbroken supersymmetries.

3 Embedding the a -model solutions in $N = 4(8)$ supergravity

We are ultimately interested in the embedding of the extreme (multi-) black-hole solutions of the a -model into $N = 8, D = 4$ supergravity, which is equivalent (upon dimensional reduction) to any of the $N = 2, D = 10$ supergravities. For simplicity, we will focus on the Neveu-Schwarz-Neveu-Schwarz (NS-NS) subsector of this theory, which can be obtained by simply setting to zero all the Ramond-Ramond (R-R) fields. As explained in Ref. [35] and in Appendix B, where more details can be found with the explicit example of the type IIA theory, this truncation is consistent (*i. e.* any solution of the truncated theory is automatically a solution of the original one) and the bosonic sector of the truncated theory is the bosonic sector of the $N = 1, D = 10$ supergravity theory.

The consistency of the truncation has a stringy explanation: the only sources for R-R fields have to be R-R fields. Therefore, there are no purely NS-NS terms in the equations of motion of the R-R fields and all terms simultaneously vanish, leaving no constraints.

Dimensional reduction of $N = 1, D = 10$ supergravity to $D = 4$ gives $N = 4, D = 4$ supergravity coupled to six matter (vector) supermultiplets [36]. Therefore, solutions of $N = 4, D = 4$ supergravity coupled to six vector supermultiplets can be considered simultaneously as solutions of $N = 8, D = 4$ (or $N = 2, D = 10$) supergravity. Since the supersymmetry transformation rules are much simpler in ten dimensions, we will uplift any solution of the $N = 4(+6V), D = 4$ theory to ten dimensions to get solutions of the $N = 1, 2$ theories.

In this section we want to identify further *consistent* truncations of the $N = 4(+6V), D = 4$ theory that lead us to the a -model for some values of a , so, in the end, and following the logic of the previous paragraph, we have a solution of the $N = 1(2), D = 10$ theory for each solution of the a -model.

To study the consistency of the truncations, we need the equations of motion. They could be derived from the action Eq. (96). However, since the a -model makes sense only in the canonical metric, we first rescale the string metric $g_{\mu\nu}$ in Eq. (96) to the canonical metric $\tilde{g}_{\mu\nu} = e^{-2\phi} g_{\mu\nu}$ and get

$$\begin{aligned} & \tilde{G}_{\alpha\beta} + 2T_{\alpha\beta}^\phi + \frac{9}{4}T_{\alpha\beta}^B \\ & -\frac{1}{4}[\partial_\alpha G_{mn}\partial_\beta G^{mn} - \frac{1}{2}\tilde{g}_{\alpha\beta}\partial_\mu G_{mn}\partial^\mu G^{mn}] \\ & -\frac{1}{4}G^{mn}G^{pq}[\partial_\alpha B_{mp}\partial_\beta B_{nq} - \frac{1}{2}\tilde{g}_{\alpha\beta}\partial_\mu B_{mp}\partial^\mu B_{nq}] \\ & +\frac{1}{2}G_{mn}\left[F^{(1)m}{}_\alpha{}^\mu F^{(1)m}{}_{\beta\mu} - \frac{1}{4}\tilde{g}_{\alpha\beta}F^{(1)m}{}_{\mu\nu}F^{(1)m\mu\nu}\right] \\ & +\frac{1}{2}G^{mn}\left[\mathcal{F}_{m\alpha}{}^\mu\mathcal{F}_{n\beta\mu} - \frac{1}{4}\tilde{g}_{\alpha\beta}\mathcal{F}_{m\mu\nu}\mathcal{F}_n{}^{\mu\nu}\right] = 0, \quad (11) \end{aligned}$$

$$\tilde{\nabla}^2\phi + \frac{3}{4}e^{-4\phi}H^2 + \frac{1}{8}e^{-2\phi}\left[G_{mn}F^{(1)m}F^{(1)n} + G^{mn}\mathcal{F}_m\mathcal{F}_n\right] = 0, \quad (12)$$

$$\begin{aligned} & \tilde{\nabla}^2 G^{rs} - G^{m(r}G^{s)n}G^{pq}[\partial G_{mp}\partial G_{nq} + \partial B_{mp}\partial B_{nq}] \\ & +\frac{1}{2}e^{-2\phi}\left[F^{(1)r}F^{(1)s} - G^{m(r}G^{s)n}\mathcal{F}_m\mathcal{F}_n\right] = 0, \quad (13) \end{aligned}$$

⁴This was pointed out to us by Paul Townsend.

$$\hat{\nabla}_\mu (G^{mr} G^{qs} \partial^\mu B_{nq}) + e^{-2\phi} \mathcal{F}_m G^{m[s} F^{(1)r]} = 0 \quad (14)$$

$$\hat{\nabla}_\mu (e^{-2\phi} G_{mn} F^{(1)n\mu\alpha}) = 0, \quad (15)$$

$$\hat{\nabla}_\mu (e^{-2\phi} G^{mn} \mathcal{F}_n^{\mu\alpha}) = 0, \quad (16)$$

$$\hat{\nabla}_\mu (e^{-4\phi} H^{\mu\alpha\beta}) = 0, \quad (17)$$

where $T_{\alpha\beta}^\phi$ is the energy-momentum tensor of ϕ (just as in Eq. (3)) and $T_{\alpha\beta}^B$ is the energy-momentum tensor of the axion two-form $B_{\alpha\beta}$

$$T_{\alpha\beta}^B = H_\alpha^{\mu\nu} H_{\beta\mu\nu} - \frac{1}{6} \tilde{g}_{\alpha\beta} H^2. \quad (18)$$

We also have to satisfy the following Bianchi identities

$$\begin{aligned} \partial F^{(1)m} &= 0, \quad \partial H = \frac{1}{2} F^{(1)m} F^{(2)}_m, \\ \partial F^{(2)}_m &= 0. \end{aligned} \quad (19)$$

Any truncation leading to the a -model must necessarily have no axion field and no axion field-strength. A reasonable choice is, then

$$\begin{aligned} B_{\alpha\beta} &= 0, & H_{\alpha\beta\gamma} &= 0, \\ G_{mn} &= -e^{2\rho_m} \delta_{mn}, & B_{mn} &= 0. \end{aligned} \quad (20)$$

Substituting into the above equations of motion and Bianchi identities we get the following equations of motion

$$\begin{aligned} \tilde{G}_{\alpha\beta} + 2T_{\alpha\beta}^\phi + \sum_m T_{\alpha\beta}^{\rho_m} \\ - \frac{1}{2} \sum_m e^{-2(\phi-\rho_m)} T_{\alpha\beta}^{(1)m} - \frac{1}{2} \sum_m e^{-2(\phi+\rho_m)} T_{m\alpha\beta}^{(2)} = 0, \end{aligned} \quad (21)$$

$$\tilde{\nabla}^2 \phi - \frac{1}{8} \sum_m e^{-2(\phi-\rho_m)} \left(F^{(1)m} \right)^2 - \frac{1}{8} \sum_m e^{-2(\phi+\rho_m)} \left(F^{(2)}_m \right)^2 = 0, \quad (22)$$

$$\tilde{\nabla}^2 \rho_m + \frac{1}{4} e^{-2(\phi-\rho_m)} \left(F^{(1)m} \right)^2 - \frac{1}{4} e^{-2(\phi+\rho_m)} \left(F^{(2)}_m \right)^2 = 0, \quad (23)$$

$$\tilde{\nabla}_\mu \left(e^{-2(\phi-\rho_m)} F^{(1)m\mu\alpha} \right) = 0, \quad (24)$$

$$\tilde{\nabla}_\mu \left(e^{-2(\phi+\rho_m)} F^{(2)}_m{}^{\mu\alpha} \right) = 0, \quad (25)$$

and the following constraints

$$F^{(1)r}{}_{\mu\nu} F^{(1)s\mu\nu} - e^{-2(\rho_r+\rho_s)} F^{(2)}_{r\mu\nu} F^{(2)}_{s\mu\nu} = 0, \quad \forall r \neq s, \quad (26)$$

$$F^{(1)r}{}_{\mu\nu} F^{(2)}_s{}^{\mu\nu} - e^{-2(\rho_r-\rho_s)} F^{(1)s}{}_{\mu\nu} F^{(2)}_s{}^{\mu\nu} = 0, \quad \forall r \neq s, \quad (27)$$

$$\sum_m F^{(1)m}{}_{\mu\nu} {}^\star F^{(2)}_{m\mu\nu} = 0. \quad (28)$$

The origins of the first two constraints are the equations of motion of the vanishing fields. The last constraint is a consistency condition between $B_{\alpha\beta} = 0$ and $H_{\alpha\beta\gamma} = 0$ due to the Bianchi identity of $B_{\alpha\beta}$ (19) and it simply means that the Chern-Simons term in the definition of H locally vanishes.

The a -model has only one scalar and one vector field. It is necessary, then, to get to it, to make an ansatz of this kind:

$$\begin{aligned} \vec{F}^{(1)} &= \vec{n}F + \vec{p} {}^\star F, & \rho_m &= c_m \varphi, \\ \vec{F}^{(2)} &= \vec{m}F + \vec{q} {}^\star F, & \phi &= b\varphi, \end{aligned} \quad (29)$$

where b and the c_m 's are constants and the vectors $\vec{n}, \vec{p}, \vec{m}, \vec{q}$ can be functions of φ and we have arranged the vector field-strengths in column vectors. F is a purely electric or magnetic vector field-strength (for definiteness we take it to be electric).

It is clear that, after the truncation Eqs. (20), no other ansatz will take us to the a -model or, at least, to the static electric black-hole solutions of the a -model.

Substituting our ansatz into the equations of motion and constraints it is possible to prove (after a considerable amount of work) that only the cases $a = \sqrt{3}, 1, 1/\sqrt{3}, 0$ can be obtained. Only in these cases can the a -model be embedded in the truncation of the $N = 4(+6V), D = 4$ supergravity theory that we have proposed. One also finds that, up to *heterotic* duality rotations, there is a very small number of ways to do this embedding in each case (see Table 1 for a complete collection of these embeddings). Before we explain our results in each case let us say that T duality acts in our truncated model by rotations separately in the space of the $A^{(1)m}$ vectors and in the space of $A^{(2)}_m$ and by the transformation (Buscher's T duality transformation [37])

$$\begin{aligned} A^{(1)m'} &= -A^{(2)}_m, \quad \rho'_m = -\rho_m, \\ A^{(2)'}_m &= -A^{(1)m}. \end{aligned} \quad (30)$$

There are also electric-magnetic dualities, which are essentially those of the a -model Eqs. (8).

All these *heterotic* dualities (which do not assume the existence of isometries in the four-dimensional solutions, but make use of the fact that, as ten-dimensional solutions they do have a six-dimensional Abelian isometry group) are just non-compact symmetries of the supergravity theories [38] and are, as explained in the Introduction, consistent with supersymmetry in the sense that a configuration and its duals have the same number of four-dimensional unbroken supersymmetries⁵ [6, 7, 5, 9, 8]. This fact allows us to study just one configuration and not its *heterotic* duals.

As we are going to see (see also [33]) the number of inequivalent extreme black-hole solutions is very small and one always can choose a representative in the equivalence class which has a maximum of two vector and two scalar fields different from zero.

3.1 $a = \sqrt{3}$ embeddings

It is easy to see that setting

$$F^{(1)} = \sqrt{2}F, \quad \phi = \frac{1}{\sqrt{3}}\varphi, \quad \rho_1 = -\frac{2}{\sqrt{3}}\varphi, \quad (31)$$

the equations of motion of the $N = 4(+6V), D = 4$ theory Eqs. (11)-(17) reduce to those of the a -model with $a = \sqrt{3}$ [25]. Taking then the multi-black-hole solution in Eqs. (5) for this value of a we get the following corresponding solution of $N = 4(+6V), D = 4$ in the string frame entirely expressed in terms of the harmonic function V :

⁵This is not necessarily true for the ten-dimensional supersymmetries [5]. If the ten-dimensional Killing spinor depends on a compact dimension, there will not be a corresponding four-dimensional Killing spinor depending only on the four non-compact space-time dimensions. This can happen even though all fields are assumed to be independent of the compact dimensions. Thus, one could find that the same solution, as a ten-dimensional solution has more solutions than as a four-dimensional solution. However, we will not find this kind of ten-dimensional Killing spinors here.

$$\left\{ \begin{aligned} ds^2 &= V^{-1}dt^2 - d\vec{x}^2, \\ e^{2\phi} &= V^{-\frac{1}{2}}, \\ G_{11} &= -V, \\ A^{(1)1}_t &= n V^{-1}. \end{aligned} \right. \quad (32)$$

Using Eqs. (98) of Appendix B we can readily express it in ten-dimensional form:

$$\left\{ \begin{aligned} d\hat{s}^2 &= V^{-1}dt^2 - d\vec{x}^2 - \left(V^{\frac{1}{2}}dx^4 + n V^{-\frac{1}{2}}dt\right)^2 - dx^L dx^L, \\ \hat{B} &= \hat{\phi} = 0. \end{aligned} \right. \quad (33)$$

This configuration is purely gravitational in ten dimensions and corresponds to the Kaluza-Klein black hole, first identified as a solution of heterotic string theory in the last of the references in [25]. Its T dual is known as the (electrically charged) H -monopole [25].

$$\left\{ \begin{aligned} d\hat{s}^2 &= V^{-1}dt^2 - d\vec{x}^2 - V^{-1}(dx^4)^2 - dx^L dx^L, \\ \hat{B} &= -n \left(V^{-\frac{1}{2}}dt\right) \wedge \left(V^{-\frac{1}{2}}dx^4\right), \\ e^{2\hat{\phi}} &= V^{-1}, \end{aligned} \right. \quad (34)$$

and has both dilaton and axion non-vanishing.

3.2 $a = 1$ embeddings

Setting

$$\phi = \varphi, \quad F^{(1)1} = F, \quad F^{(2)}_1 = \mp F, \quad (35)$$

we get the $a = 1$ model. The corresponding $N = 4, D = 4$ solution in the string frame is

$$\left\{ \begin{array}{lcl} ds^2 & = & V^{-2} dt^2 - d\vec{x}^2, \\ e^{2\phi} & = & V^{-1}, \\ A^{(1)1}_t = \mp A^{(2)}_{1t} & = & n V^{-1}, \end{array} \right. \quad (36)$$

and the corresponding $N = 1(2), D = 10$ solution is

$$\left\{ \begin{array}{lcl} d\hat{s}^2 & = & V^{-2} dt^2 - d\vec{x}^2 - (dx^4 + n V^{-1} dt)^2 \\ & & - dx^L dx^L, \\ \hat{B} & = & \mp n V^{-1} dt \wedge (dx^4 + n V^{-1} dt), \\ e^{2\hat{\phi}} & = & V^{-1}. \end{array} \right. \quad (37)$$

If we choose the minus sign, as explained in Appendix B, the matter vector field combination $F^{(1)1} + \mathcal{F}_1 = F^{(1)1} + F^{(2)}_1$ vanishes, and only the supergravity vector field combination $F^{(1)1} - \mathcal{F}_1 = F^{(1)1} - F^{(2)}_1$ remains. The $a = 1$ model can thus be embedded in the pure $N = 4, D = 4$ supergravity theory. This was the embedding proposed in Ref. [11], and, as we will see in the next section, it is the embedding which admits Killing spinors and unbroken $N = 4, D = 4$ supersymmetry (one half of it).

If we choose the plus sign, only the matter vector field combination remains, and the resulting embedding does not have any $N = 4(+6V), D = 4$ unbroken supersymmetry [13].

This result seems paradoxical, since, after all, the four-dimensional solutions are identical, and Bogomol'nyi-type bounds must be saturated in both cases.

All this was done in the framework of the $N = 4(+6V), D = 4$ theory which results from dimensional reduction of the *positive chirality* $N = 1, D = 10$ supergravity theory, which is the standard choice. In $N = 8$ supergravity we are forced to consider both chiralities and the apparent paradox will be explained (see next section).

On top of the two embeddings (35) there is another embedding of the $a = 1$ multi-black-hole solution [16]:

$$F^{(1)1} = F, \quad F^{(1)2} = \pm e^{-2\varphi} F, \quad \rho_1 = -\varphi, \quad \rho_1 = +\varphi. \quad (38)$$

The corresponding $N = 4(+6V), D = 4$ solution in the string frame is

$$\left\{ \begin{array}{lcl} ds^2 & = & V^{-1} dt^2 - V d\vec{x}^2, \\ G_{11} & = & -V, \\ G_{22} & = & -V^{-1}, \\ A^{(1)1}_t & = & n V^{-1}, \\ A^{(1)2}_{\underline{i}} & = & m V_{\underline{i}}, \end{array} \right. \quad (39)$$

and the ten-dimensional solution is

$$\left\{ \begin{array}{lcl} d\hat{s}^2 & = & V^{-1} dt^2 - V d\vec{x}^2 - \left(V^{\frac{1}{2}} dx^4 + n V^{-\frac{1}{2}} dt \right)^2 \\ & & - \left(V^{-\frac{1}{2}} dx^5 + m V^{-\frac{1}{2}} V_{\underline{i}} dx^{\underline{i}} \right)^2 - dx^L dx^L, \\ \hat{B} & = & \hat{\phi} = 0. \end{array} \right. \quad (40)$$

3.3 $a = 1/\sqrt{3}$ embeddings

Setting

$$\begin{aligned} F^{(1)1} &= \sqrt{\frac{2}{3}} F, & F^{(1)2} &= \mp F^{(2)}_2 = \sqrt{\frac{2}{3}} e^{-\frac{2}{\sqrt{3}}\varphi} F, \\ \phi &= -\frac{1}{\sqrt{3}}\varphi, & \rho_1 &= -\frac{2}{\sqrt{3}}\varphi, \end{aligned} \quad (41)$$

we get the $a = 1/\sqrt{3}$ model and the corresponding solution of the $N = 4(+6V), D = 4$ theory⁶

⁶Observe that the Hodge dual $\star F$ in the above formulae has to be calculated in the Einstein frame metric.

$$\left\{ \begin{array}{lcl} ds^2 & = & V^{-1}dt^2 - V^2d\vec{x}^2, \\ e^{2\phi} & = & V^{\frac{1}{2}}, \\ G_{11} & = & -V, \\ A^{(1)1}_t & = & n V^{-1}, \\ A^{(1)2}_{\underline{i}} & = & \mp A^{(2)}_{2\underline{i}} = nV_{\underline{i}}, \end{array} \right. \quad (42)$$

where $V_{\underline{i}}$ satisfies the equation

$$\partial_{[\underline{i}}V_{\underline{j}]} = \frac{1}{2}\epsilon_{ijk}\partial_{\underline{k}}V. \quad (43)$$

The corresponding solution of the $N = 1(2), D = 10$ theory is

$$\left\{ \begin{array}{lcl} d\hat{s}^2 & = & V^{-1}dt^2 - V^2d\vec{x}^2 - \left(V^{\frac{1}{2}}dx^4 + n V^{-\frac{1}{2}}dt\right)^2 \\ & & - (dx^5 + n V_{\underline{i}}dx^{\underline{i}})^2 - dx^L dx^L, \\ \hat{B} & = & \mp n V_{\underline{i}}dx^{\underline{i}} \wedge (dx^5 + n V_{\underline{i}}dx^{\underline{i}}), \\ e^{2\hat{\phi}} & = & V. \end{array} \right. \quad (44)$$

In the framework explained at the beginning of this section, no other embedding of the $a = 1/\sqrt{3}$ multi-black-hole solution is possible.

3.4 $a = 0$ embeddings

Setting

$$\begin{aligned} F^{(1)1} &= \frac{1}{2}F, & F^{(1)2} &= \frac{1}{2}\star F, \\ F^{(2)}_1 &= \mp \frac{1}{2}F, & F^{(2)}_2 &= \mp \frac{1}{2}\star F, \end{aligned} \quad (45)$$

one gets the $a = 0$ model (Einstein-Maxwell theory). The solution of the $N = 4(+6V), D = 4$ theory is

$$\left\{ \begin{array}{lcl} ds^2 & = & V^{-2}dt^2 - V^2d\vec{x}^2, \\ A^{(1)1}_t = \mp A^{(2)}_{1t} & = & n V^{-1}, \\ A^{(1)2}_{\underline{i}} = \mp A^{(2)}_{2\underline{i}} & = & m V_{\underline{i}}, \end{array} \right. \quad (46)$$

and uplifted to ten dimensions is

$$\left\{ \begin{array}{lcl} d\hat{s}^2 & = & V^{-2}dt^2 - V^2d\vec{x}^2 - (dx^4 + n V^{-1}dt)^2 \\ & & - (dx^5 + m V_{\underline{i}}dx^{\underline{i}})^2 - dx^L dx^L, \\ \hat{B} & = & \mp n V^{-1}dt \wedge (dx^4 + n V^{-1}dt) \\ & & \mp m V^{-1}V_{\underline{i}}(Vdx^{\underline{i}}) \wedge (dx^5 + m V_{\underline{i}}dx^{\underline{i}}), \end{array} \right. \quad (47)$$

With the minus sign, this is the standard embedding of the extreme Reissner-Nordström solution into $N = 4(+6), D = 4$ supergravity [11]. We can repeat here the discussion we made in the $a = 1$ case with respect to the relative sign of $F^{(1)m}$ and $F^{(2)}_m$ and the matter or supergravity nature of the vector fields.

No other purely electric or magnetic embeddings exist (up to dualities), but there exist some other (dyonic) embeddings [14] that we are going to discuss now⁷.

3.4.1 Dyonic embeddings

The simplest of these embeddings is the following:

$$F^{(1)1} = F \pm \star F. \quad (48)$$

The essential property that makes this embedding a solution of the $N = 4(+6V), D = 4$ theory is that, given that F is purely electric or magnetic $F\star F = 0$ and, then $(F^{(1)1})^2 = 0$. The solution is, thus,

$$\left\{ \begin{array}{lcl} ds^2 & = & V^{-2}dt^2 - V^2d\vec{x}^2, \\ A^{(1)1} & = & \sqrt{2}(n V^{-1}dt + m V_{\underline{i}}dx^{\underline{i}}), \end{array} \right. \quad (49)$$

⁷Observe that for the other values of a no dyonic embedding of any kind exists. This reflects two facts: (i) there are no dyonic solution of the a -model for $a \neq 0, 1$, and (ii) the dyonic solution of the $a = 1$ model is clearly not a solution of the $N = 4, D = 4$ theory, as explained in Ref. [11]

and the corresponding ten-dimensional solution is

$$\begin{cases} d\hat{s}^2 &= V^{-2}dt^2 - V^2 d\vec{x}^2 - [dx^4 + \sqrt{2}(n V^{-1}dt + m V_i dx^i)]^2 \\ &- dx^L dx^L, \\ \hat{B} &= \hat{\phi} = 0. \end{cases} \quad (50)$$

This embedding can be generalized to the form

$$\vec{F}^{(1)} = \vec{n}(F \pm *F), \quad \vec{F}^{(2)} = \vec{m}(F \mp *F), \quad (51)$$

provided $n^r n^r + m^r m^r = 1$ and $n^r m^s = n^s m^r$. Some of these embeddings are just T or S duals of the simplest one, but we will not pursue this problem here.

4 Unbroken $N = 4(8)$ supersymmetries of the a -model solutions

In the previous section we have found solutions of $N = 1(2), D = 10$ supergravity which in four dimensions correspond to the multi-black-hole solutions of the a -model. In this section we are going to find the unbroken supersymmetries of the ten-dimensional solutions, which is a way of finding the unbroken supersymmetries of the four dimensional solutions in $N = 4(8)$ supergravity.

As explained in Appendix B, when the R-R fields of the type IIA theory vanish, the supersymmetry rules reduce to Eqs. (94) that we rewrite here for convenience

$$\begin{aligned} \delta_\epsilon \hat{\psi}_a^{(\pm)} &= \hat{\nabla}_a^{(\pm)} \hat{\epsilon}^{(\pm)}, \\ \delta_\epsilon \hat{\lambda}^{(\pm)} &= \left(\not{\partial} \hat{\phi} \pm \frac{1}{4} \hat{H} \right) \hat{\epsilon}^{(\pm)}, \end{aligned}$$

where $\hat{\nabla}_a^{(\pm)}$ are the covariant derivatives associated to the two torsionful spin connections

$$\hat{\Omega}_{\hat{a}\hat{b}\hat{c}}^{(\pm)} = \hat{\omega}_{\hat{a}\hat{b}\hat{c}} \mp \frac{3}{2} \hat{H}_{\hat{a}\hat{b}\hat{c}}.$$

Taking just the positive chirality (upper signs) in the above equations one gets the supersymmetry rules of the conventional $N = 1, D = 10$ supergravity theory $N = 1^{(+)}, D = 10$. The other sign choice gives the supersymmetry rules of another $N = 1, D = 10$ theory ($N = 1^{(-)}, D = 10$) that can be constructed.

In fact, to find unbroken supersymmetries, in many cases we can use the symmetry existing between the two chirality sectors of the NS-NS sector of type IIA supergravity: “ C duality”. A C duality transformation changes the chirality of the spinors and the sign of the axion and leaves the NS-NS sector of the type IIA theory invariant. However, from the point of view of the $N = 1$ theories, C duality is not a symmetry. Each theory has a definite chirality that cannot be changed. Instead, C duality takes us from the $N = 1^{(+)}, D = 10$ theory to the $N = 1^{(-)}, D = 10$ theory and is another (very simple) example of string/string duality.

Then, when we have two embeddings which differ by just the sign of the axion, we can use C duality arguments to translate the results of one chirality sector to the other.

The necessary ingredients to find the unbroken supersymmetries are just the dilaton, the axion field strength and the spin connection coefficients, which are calculated for the kind of metric we are dealing with in Appendix C.

4.1 Unbroken supersymmetries of the $a = \sqrt{3}$ embeddings

The supersymmetry equations for $a = \sqrt{3}$ black holes are the most straightforward. Consider the field configuration of the electrically charged Kaluza-Klein black hole described in Section 3.1. Then from $\hat{\phi} = \hat{H} = 0$, it follows that the supersymmetry rules are identical for both positive and negative chirality ten-dimensional spinors and that the dilatino equation is trivially satisfied. Following Appendix B, it is straightforward to show that the Killing spinors of this embedding are those which satisfy

$$\hat{\epsilon}^{(\pm)} = V^{-\frac{1}{4}} \hat{\epsilon}_{(0)}^{(\pm)}, \quad (52)$$

$$\hat{\gamma}_0 \hat{\gamma}_4 \hat{\epsilon}_{(0)}^{(\pm)} = -n \hat{\epsilon}_{(0)}^{(\pm)}, \quad (53)$$

where $\hat{\epsilon}_{(0)}^{(\pm)}$ is a constant spinor. This chirality condition on the subspinor in the (04) sector implies that precisely half of the supersymmetries are broken, for either positive or negative ten-dimensional spinor. So half of the supersymmetries are preserved for each of the opposite chirality $N = 1$ theories in $D = 10$, and, as a result, half of each of the corresponding $N = 4$ supersymmetries in $D = 4$, and, therefore, a total of a half of the $N = 8, D = 4$ supersymmetries.

Had we performed the same calculation for the T dual of this solution, namely the electrically charged H -monopole, which has a non-vanishing axion field strength and is not C duality invariant, we would have gotten a similar result:

$$\hat{\epsilon}^{(\pm)} = V^{-\frac{1}{4}} \epsilon_{(0)}^{(\pm)}, \quad (54)$$

$$\hat{\gamma}_0 \hat{\gamma}_4 \hat{\epsilon}_{(0)}^{(\pm)} = \mp n \epsilon_{(0)}^{(\pm)}. \quad (55)$$

As can be seen above, the Killing spinors are invariant under T duality in the positive chirality sector, but are only covariant in the negative chirality sector. This is due to the different ways in which the two torsionful spin connections $\hat{\Omega}^{(\pm)}$ transform [39]. The number of $N = 4(8), D = 4$ unbroken supersymmetries does not change, though. It would also be the same had we taken the magnetically charged S dual versions of both the Kaluza-Klein and H -monopole, although in these latter two cases, the chirality condition is imposed on the (1234) sector of the spinor. Henceforth we will not consider explicitly the S or T dual versions of these solutions.

4.2 Unbroken supersymmetries of the $a = 1$ embeddings

The situation for the $a = 1$ embeddings is a bit more subtle. Consider the first embedding in Section 3.2, Eqs. (35-37) and the positive (conventional) chirality $N = 1^{(+)}, D = 10$ theory. When the minus sign is chosen, both the gravitino and dilatino equations reduce to the same conditions, namely

$$\hat{\epsilon}^{(+)} = V^{-\frac{1}{2}} \epsilon_{(0)}^{(+)}, \quad (56)$$

$$\hat{\gamma}_0 \hat{\gamma}_4 \hat{\epsilon}_{(0)}^{(+)} = -n \epsilon_{(0)}^{(+)}. \quad (57)$$

When the plus sign is chosen, there are no Killing spinors in the $N = 1^{(+)}, D = 10$ theory (*i.e.* $\hat{\epsilon}^{(+)} = 0$ is the only consistent solution).

As explained above, these two choices of sign correspond respectively to declaring that the four dimensional vector is a supergravity vector (and that the matter vector vanishes) in the $N = 4, D = 4$ theory and declaring exactly the opposite. We have just reproduced the results of Refs. [11, 13] respectively, although in a different setting.

We can now use C duality to find the unbroken supersymmetries in the $N = 1^{(-)}, D = 10$ theory. For the minus sign choice there is now no Killing spinor, and for the plus sign one gets the same condition as for the minus sign in the $N = 1^{(+)}, D = 10$ theory, namely

$$\hat{\epsilon}^{(-)} = V^{-\frac{1}{2}} \epsilon_{(0)}^{(-)}, \quad (58)$$

$$\hat{\gamma}_0 \hat{\gamma}_4 \hat{\epsilon}_{(0)}^{(-)} = -n \epsilon_{(0)}^{(-)}. \quad (59)$$

Then, both choices of sign (both embeddings) are supersymmetric in one sector, and in that sector a half of the $N = 4, D = 4$ supersymmetries are unbroken, just as in the $a = \sqrt{3}$ case. Since we are forced to consider both sectors, the total number of $N = 8$ unbroken supersymmetries is the same for both choices: $1/4$. This result, which resolves the paradox, could also be explained by the fact that there are no matter vector fields in the $N = 8$ theory: *all* vectors are supergravity vector fields.

Finally, consider the embedding of Eqs. (38-40). This case, as the $a = \sqrt{3}$ (Kaluza-Klein) case, is C duality symmetric, and the supersymmetry transformations are identical for both chiralities. In addition to the conditions

$$\hat{\epsilon}^{(\pm)} = V^{-\frac{1}{4}} \epsilon_{(0)}^{(\pm)}, \quad (60)$$

$$\hat{\gamma}_0 \hat{\gamma}_4 \hat{\epsilon}_{(0)}^{(\pm)} = -n \epsilon_{(0)}^{(\pm)}, \quad (61)$$

we get the condition

$$\hat{\Gamma}_5 \hat{\epsilon}^{\pm} = -m \epsilon^{\pm}, \quad (62)$$

coming from the magnetic sector, where $\hat{\Gamma}_5 = \hat{\gamma}_1 \hat{\gamma}_2 \hat{\gamma}_3 \hat{\gamma}_5$. As a result, the supersymmetries already halved by the first condition are halved again in each sector. This implies that for each $N = 4$ theory, one quarter of the supersymmetries are preserved [16, 17]. Therefore, again, but in a different fashion, for the $N = 8$ theory, 2 supersymmetries (one quarter) are preserved.

4.3 Unbroken supersymmetries of the $a = 1/\sqrt{3}$ embeddings

In some sense the $a = 1/\sqrt{3}$ black hole is a combination of an $a = 1$ black hole with a dual-charged $a = \sqrt{3}$ black hole. Again following Appendix C, it is straightforward to show that for the minus sign choice in the embedding of Eqs. (41-44)

$$\hat{\epsilon}^{(\pm)} = V^{-\frac{1}{4}} \epsilon_{(0)}^{(\pm)}, \quad (63)$$

$$\hat{\Gamma}_5 \hat{\epsilon}_{(0)}^{(+)} = -n \epsilon_{(0)}^{(+)}, \quad (64)$$

$$\hat{\gamma}_0 \hat{\gamma}_4 \hat{\epsilon}_{(0)}^{(+)} = -n \hat{\epsilon}_{(0)}^{(+)} . \quad (65)$$

and $\hat{\epsilon}^{(-)} = 0$. This implies that 1/4 of the positive chirality $N = 4$ supersymmetries (i.e. the supersymmetries arising from the reduction of the positive chirality ten-dimensional spinor) are preserved while none of the negative chirality supersymmetries are preserved. As a result, only one of the $N = 8$ supersymmetries is preserved.

For the plus sign choice in Eqs. (41-44), none of the positive chirality supersymmetries are preserved, while one of the four negative chirality supersymmetries is preserved, explicitly

$$\hat{\epsilon}^{(-)} = V^{-\frac{1}{4}} \hat{\epsilon}_{(0)}^{(-)} , \quad (66)$$

$$\hat{\Gamma}_5 \hat{\epsilon}_{(0)}^{(-)} = +n \hat{\epsilon}_{(0)}^{(-)} , \quad (67)$$

$$\hat{\gamma}_0 \hat{\gamma}_4 \hat{\epsilon}_{(0)}^{(-)} = +n \hat{\epsilon}_{(0)}^{(-)} . \quad (68)$$

A simple way of seeing this pattern is as follows: the configuration described in Eqs. (41-44) represent a combination of a magnetic $a = 1$ black hole and an electric $a = \sqrt{3}$ black hole.⁸ The $a = 1$ part of the configuration preserves half the supersymmetries for one chirality and none for the other. The $a = \sqrt{3}$ part then independently halves again whatever remaining supersymmetries exist in each sector. As a result, we are left with only an eighth of the $N = 8$ supersymmetries.

4.4 Unbroken supersymmetries of the $a = 0$ embeddings

The $a = 0$ embeddings described in Section 3.4, Eqs. (45-47) have precisely the same supersymmetry breaking pattern as the $a = 1/\sqrt{3}$ solutions described above, with the difference that an $a = 0$ solution can most usefully be thought of as a combination of two $a = 1$ black holes, one electric and one magnetic, each imposing an independent chirality condition.

The dyonic embeddings we have shown, by contrast, break all of the spacetime supersymmetries. These embeddings do not, however, exhaust all possible dyonic embeddings. This can be seen by noting that there exist dyonic ERN black holes which preserve some supersymmetry in certain $N = 2$ truncations⁹.

⁸It is interesting that this composite viewpoint is consistent with the bound state picture in [33] at the level of the Killing spinor equations.

⁹The purely electric and purely magnetic ERN black holes are supersymmetric in $N =$

5 Conclusion

In this paper we have determined which four-dimensional extreme dilaton black-hole solutions can be embedded in $N = 4$ supergravity, for which values of the parameter a and in how many inequivalent ways this can be done (that is, not related by duality symmetries). We have also studied the $N = 4$ unbroken supersymmetries of these black holes as well as their $N = 8$ unbroken supersymmetries, making use of the fact that the $N = 4$ theory can be considered as a consistent truncation of the $N = 8$. Our results are summarized in Table 1.

We have found that only the $a = \sqrt{3}, 1, 1/\sqrt{3}, 0$ dilaton black holes can be embedded in the $N = 4$ theory and that this can be done in a very limited number of inequivalent ways (not related by T or S duality). There is only one embedding of the $a = \sqrt{3}$ dilaton black hole, three of the $a = 1$ one and two of the $a = 1/\sqrt{3}$. The $a = 0$ can be embedded in just two different (purely electric or magnetic) ways, but other (dyonic) embeddings are possible. All the inequivalent embeddings in the $N = 4$ theory have different amounts of unbroken supersymmetry.

The situation changes when we consider the embeddings in the $N = 8$ theory: *all embeddings of the same dilaton black hole are equivalent under U duality and have the same number of unbroken supersymmetries* with the exception of the dyonic embedding of the $a = 0$ extreme black hole. There are U duality transformations that relate embeddings which are inequivalent in the $N = 4$ theory and do not change the number of $N = 8$ unbroken supersymmetries but do change the number of $N = 4$ unbroken supersymmetries, essentially by shifting the unbroken supersymmetries from one chirality sector to the other. One example is the C duality transformation that interchanges the two chirality sectors and supergravity and matter fields of a given $N = 4$ theory (all vectors are supergravity vectors in the $N = 8$ theory and this is why C duality is a symmetry of this theory).

Note that our analysis applies to the string-like solitons constructed in [9], where some solutions were found to preserve some supersymmetry provided one made a chirality choice that matched the overall chirality of the $N = 1, D = 10$ theory from which the $N = 4$ theory was reduced. On making the opposite chirality choice, however, it was found that the solution broke all supersymmetries. This also presented an apparent paradox, since both embeddings represent essentially identical solutions with analogous Bogomol'nyi bounds. From the results in this paper, however, it follows immediately that the “wrong” chirality choice embedding simply corresponds to a solution which preserves supersymmetry in

$2, D = 4$ supergravity. This theory has an electric-magnetic duality symmetry that preserves unbroken supersymmetries (see, for instance, the second lecture in Ref. [10]), and, therefore, the dyonic ERN black hole is supersymmetric in that theory, which can be obtained by a consistent truncation of $N = 8, D = 4$ supergravity.

a	ϕ	ρ_1	ρ_2	$F^{(1)1}$	$F^{(2)}_1$	$F^{(1)2}$	$F^{(2)}_2$	(n_+, n_-)
$\sqrt{3}$	$\frac{1}{\sqrt{3}}\varphi$	$-\frac{2}{\sqrt{3}}\varphi$	0	$\sqrt{2}F$	0	0	0	$(\frac{1}{2}, \frac{1}{2})$
1	φ	0	0	F	$-F$	0	0	$(\frac{1}{2}, 0)$
	φ	0	0	F	$+F$	0	0	$(0, \frac{1}{2})$
	0	$-\varphi$	φ	F	0	$e^{-2\varphi}F$	0	$(\frac{1}{4}, \frac{1}{4})$
$\frac{1}{\sqrt{3}}$	$-\frac{1}{3}\varphi$	$-\frac{2}{3}\varphi$	0	$\sqrt{\frac{2}{3}}F$	0	$\sqrt{\frac{2}{3}}e^{2\phi}F$	$-\sqrt{\frac{2}{3}}e^{2\phi}F$	$(\frac{1}{4}, 0)$
	$-\frac{1}{3}\varphi$	$-\frac{2}{3}\varphi$	0	$\sqrt{\frac{2}{3}}F$	0	$\sqrt{\frac{2}{3}}e^{2\phi}F$	$+\sqrt{\frac{2}{3}}e^{2\phi}F$	$(0, \frac{1}{4})$
0	0	0	0	$\frac{1}{\sqrt{2}}F$	$-\frac{1}{\sqrt{2}}F$	$\frac{1}{\sqrt{2}}F$	$-\frac{1}{\sqrt{2}}F$	$(\frac{1}{4}, 0)$
	0	0	0	$\frac{1}{\sqrt{2}}F$	$+\frac{1}{\sqrt{2}}F$	$\frac{1}{\sqrt{2}}F$	$+\frac{1}{\sqrt{2}}F$	$(0, \frac{1}{4})$
	0	0	0	$F \pm F$	0	0	0	$(0, 0)$

Table 1: In this table we give the different embeddings (up to $N = 4$ (heterotic) dualities) of the $a = \sqrt{3}, 1, 1/\sqrt{3}, 0$ purely electric (or magnetic) solutions Eqs. (5) in $N = 4(8)$ supergravity. It is read in the following manner: if the $N = 4$ fields of the top row take the values given in the following rows, in terms of φ and F , where F is either purely electric or purely magnetic, then the $N = 4$ equations of motion reduce to those of the a -model (2) for the value of a given in the first column. In the last column we list the unbroken supersymmetry in the two $N = 4$ sectors of positive and negative chirality as a fraction of the total.

the opposite chirality $N = 1, D = 10$ theory. In the $N = 8$ theory, both chirality choices lead to embeddings which preserve the same amount of supersymmetry. This conclusion also applies to analyticity versus anti-analyticity conditions in certain $N = 1, D = 4$ truncations [9].

Furthermore, our analysis in this paper can be generalized in a straightforward manner to arbitrary supersymmetric p -branes, both isotropic and anisotropic, in arbitrary D spacetime dimensions, following the oxidation/reduction procedures discussed in [40] (see also [41]).

It is tempting to conclude that all embeddings of any given four-dimensional solution should be equivalent in the $N = 8$ theory. Previously it was thought that only special embeddings of a solution in a supergravity theory had unbroken supersymmetry. Our results seem to indicate that if a solution saturates certain bounds and there is one supersymmetric embedding, all possible embeddings will also be supersymmetric, and none of them will be singled out.

This hypothesis could explain why we have found no embeddings with $(\frac{1}{8}, \frac{1}{8})$ of unbroken supersymmetries, that is, with $\frac{1}{8}$ of the $N = 8$ supersymmetries unbroken, half of them in each chirality sector. U duality transformations can only change the number of unbroken supersymmetries by an integer number of $N = 8$ supersymmetries. Thus, if we start with the $(\frac{1}{4}, 0)$ embedding of the $a = 1/\sqrt{3}$ black hole, we can only get to the $(0, \frac{1}{4})$ embedding, by using a U duality transformation that shifts one $N = 8$ supersymmetry from the positive to the negative chirality sector. If our hypothesis is true, then, we cannot access this embedding by U duality, and it does not exist (certainly we have not found it).

However, we cannot ignore the presence of a manifest exception to this hypothesis: the dyonic embedding of the extreme Reissner-Nordström black hole. An explanation of the existence of this solution in terms of bound states is not apparent. Instead, one could hope for a larger framework in which this embedding is supersymmetric, just as embeddings which are non-supersymmetric in the $N = 4$ picture are supersymmetric in the $N = 8$ framework [14].

In calculating the $N = 8$ unbroken supersymmetries we have used the ten-dimensional type IIA theory. Since we are considering four-dimensional solutions, our results (the number of $N = 8$ unbroken supersymmetries) would be identical had we worked with the type IIB theory. It is, though, of some interest, to know what the ten-dimensional Killing spinors would look like in the type IIB case, since this theory is chiral and it has spinors of only one chirality. Now one has two sectors of the same chirality.

The chirality of the type IIB theory is conventionally positive, so one gets the positive chirality $N = 1^{(+)}$ theory upon truncation of the bosonic RR fields and one of the spinor sectors (say the second). But there is also a negative chirality type IIB theory (type IIB $^{(-)}$) characterized by the different chirality of the spinors and by the fact that the five-form $\hat{F}_{\hat{\mu}_1 \dots \hat{\mu}_5}$ (which is the field-strength

of the four-form field $\hat{D}_{\hat{\mu}_1 \dots \hat{\mu}_4}$ in the notation of Refs. [35, 42]) instead of being self-dual as in the type IIB⁽⁺⁾ theory, is anti-self-dual. The same truncation of this theory would give us the $N = 1^{(-)}$ one.

The situation is summarized in Table 2.

It takes, then, little thought to arrive at the conclusion that, since both type II theories describe the same degrees of freedom but are “arranged” in different ways, there must be a supersymmetric basis for the type IIB⁽⁺⁾ theory such that, in absence of bosonic R-R fields, the sector corresponding to the gravitino $\hat{\psi}_a^{(+1)}$ is the same as in the type IIA¹ theory and the sector corresponding to the gravitino $\hat{\psi}_a^{(+2)}$ is the same as the sector $\hat{\psi}_a^{(-1)}$ of the type IIA¹ theory. From this viewpoint, then, in the absence of R-R fields, the supersymmetry transformation rules should be

$$\begin{aligned} \delta_\epsilon \hat{\psi}_a^{(+1)} &= \hat{\nabla}_a^{(+)} \hat{\epsilon}^{(+1)}, \quad \delta_\epsilon \hat{\lambda}^{(+1)} = \left(\hat{\phi} + \frac{1}{4} \hat{H} \right) \hat{\epsilon}^{(+1)}, \\ \delta_\epsilon \hat{\psi}_a^{(+2)} &= \hat{\nabla}_a^{(-)} \hat{\epsilon}^{(+2)}, \quad \delta_\epsilon \hat{\lambda}^{(+2)} = \left(\hat{\phi} - \frac{1}{4} \hat{H} \right) \hat{\epsilon}^{(+2)}. \end{aligned} \quad (69)$$

In this case, all our results for the Killing spinors in the type IIA theory can be translated to the type IIB⁽⁺⁾ by just replacing $\hat{\epsilon}^{(+)}$ by $\hat{\epsilon}^{(+1)}$ and $\hat{\epsilon}^{(-)}$ by $\hat{\epsilon}^{(+2)}$. In the type IIB⁽⁺⁾ C duality would relate the 1 and 2 sectors $\hat{\psi}_a^{(+1)}$ and $\hat{\psi}_a^{(+2)}$ which now happen to have the same chirality. In the $N = 1$ context, C duality would relate two different $N = 1^{(+)}$ theories of the same chirality, but with different supersymmetry rules, the difference being the sign of $\hat{H}_{\hat{\alpha}\hat{\beta}\hat{\gamma}}$.

Finally, to complete the picture of all different embeddings of the same four-dimensional solutions in the $N = 8$ theory being related by U duality transformations (so that there is only one inequivalent embedding for each solution) one should also study R-R and mixed embeddings. It is, however, unlikely that the picture will change from what we have presented above, since U duality treats both sectors on the same footing and also interchanges them.

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type IIA ¹ →	$\hat{\psi}_a^{(+1)}$	$\hat{\psi}_a^{(-1)}$
type IIA ² →	$\hat{\psi}_a^{(+2)}$	$\hat{\psi}_a^{(-2)}$
	↑ type IIB ⁽⁺⁾	↑ type IIB ⁽⁻⁾

Table 2: Here we have represented symbolically the gravitini of four possible $N = 2, D = 10$ theories that we can define in such a way that the “common sectors” are in the corresponding intersection of row and column. Each common sector loosely corresponds to an $N = 1, D = 10$ theory.

A Conventions

We denote with two hats eleven-dimensional objects, with one hat, ten-dimensional objects and with no hats four-dimensional objects. Greek and underlined latin or numerical indices are always world indices, and simple latin or numerical indices are Lorentz indices. We reserve the indices i, j, k for the values 1, 2, 3. The completely antisymmetric symbol ϵ_{ijk} is defined by $\epsilon_{123} = +1$.

We use the mostly minuses signature $(+ - - \dots -)$ and work in the string-frame metric, except when we use the canonical metric that we denote by a tilde $\tilde{g}_{\mu\nu}$. The antisymmetric Levi-Civita tensor $\hat{\epsilon}$ is defined by

$$\hat{\epsilon}^{\hat{\mu}_0 \dots \hat{\mu}_{10}} = 1. \quad (70)$$

Our spin connection ω (in D dimensions) is defined by

$$\omega_\mu{}^{ab}(e) = -e^{\nu[a} \left(\partial_\mu e_\nu{}^{b]} - \partial_\nu e_\mu{}^{b]} \right) - e^{\rho[a} e^{\sigma b]} (\partial_\sigma e_{c\rho}) e_\mu{}^c. \quad (71)$$

The curvature tensor corresponding to this spin connection field is defined by

$$R_{\mu\nu}{}^{ab}(\omega) = 2\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\omega_{[\mu}{}^{ac}\omega_{\nu]}{}^b{}_c, \quad R(\omega) \equiv e^\mu{}_a e^\nu{}_b R_{\mu\nu}{}^{ab}(\omega). \quad (72)$$

Indices not shown are assumed to be completely antisymmetrized world indices. Thus, for instance,

$$\partial H = \frac{1}{2} F^{(1)m} F^{(2)}{}_m, \quad (73)$$

(the Bianchi identity for B Eq. (19) stands for

$$\partial_{[\alpha} H_{\beta\gamma\delta]} = \frac{1}{2} F^{(1)m}{}_{[\alpha\beta} F^{(2)}{}_{m\gamma\delta]} . \quad (74)$$

B The type IIA bosonic action and supersymmetry transformation rules in the string frame, their truncation to $N = 1$ and further reduction to $D = 4$

B.1 Dimensional reduction from $D = 11$ to $D = 10$

The best way to obtain the supersymmetry transformation laws of the ten-dimensional type IIA theory in the string frame is by direct dimensional reduction of $N = 1, D = 11$ supergravity, since we know that the dilaton is just a function of the only scalar modulus field that appears [4, 35].

The bosonic fields of $N = 1, D = 11$ supergravity [43] are the elfbein and a three-form potential

$$\left\{ \hat{e}_{\hat{\mu}}^{\hat{a}}, \hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}} \right\} . \quad (75)$$

The field strength of the three-form is

$$\hat{G} = \partial \hat{C} , \quad (76)$$

and the action for these bosonic fields is

$$\hat{S} = \frac{1}{2} \int d^{11}x \sqrt{\hat{g}} \left[-\hat{R} + a_1 \hat{G}^2 + a_2 \frac{1}{\sqrt{\hat{g}}} \hat{e} \hat{G} \hat{G} \hat{C} \right] . \quad (77)$$

where

$$a_1^3/a_2^2 = 2^8 3^5, a_1 > 0 . \quad (78)$$

The only fermionic field of this theory is the gravitino $\hat{\psi}_{\hat{\mu}}$, whose supersymmetry transformation law, for purely bosonic configurations is¹⁰

¹⁰Our gamma matrices are in a purely imaginary Majorana representation and have the anticommutation relations $\left\{ \hat{\gamma}^{\hat{a}}, \hat{\gamma}^{\hat{b}} \right\} = +2\hat{\eta}^{\hat{a}\hat{b}}$.

$$\frac{1}{\sqrt{2}} \delta_{\epsilon} \hat{\psi}_{\hat{\mu}} = \hat{\nabla}_{\hat{\mu}} \hat{\epsilon} - 6i \frac{a_2}{a_1} \left(\hat{\gamma}^{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}{}_{\hat{\mu}} - 8 \hat{\gamma}^{\hat{\beta}\hat{\gamma}\hat{\delta}} \delta^{\hat{\alpha}}{}_{\hat{\mu}} \right) \hat{G}_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} \hat{\epsilon} , \quad (79)$$

The dimensional reduction has been explicitly performed in Ref. [35]. We can use the same ansatz for the elfbein to get the same result for the $D = 10$ type IIA action in the string frame. We refer the reader to that reference for details on the definitions of the ten-dimensional fields and field strengths. On top of that we make the identifications

$$\hat{\gamma}^{\hat{a}} = \hat{\gamma}^{\hat{a}}, \quad \hat{a} = 0, \dots, 9, \quad (80)$$

$$\hat{\gamma}^{10} = -i \hat{\gamma}_{11} = \hat{\gamma}^0 \dots \hat{\gamma}^9, \quad (81)$$

so $\hat{\gamma}_{11}$ satisfies $(\hat{\gamma}_{11})^2 = +1$ and can be used to define ten-dimensional chiralities. We define the ten-dimensional spinors¹¹

$$\hat{\epsilon} = e^{\frac{1}{8}\hat{\phi}} \hat{\epsilon}, \quad (82)$$

$$\hat{\psi}_{\hat{a}} = \frac{1}{\sqrt{2}} e^{-\frac{1}{8}\hat{\phi}} \left(\hat{\psi}_{\hat{a}} - \frac{i}{2} \hat{\gamma}_{\hat{a}} \hat{\gamma}_{11} \hat{\psi}_{11} \right), \quad (83)$$

$$\hat{\lambda} = \frac{-3i}{\sqrt{2}} e^{-\frac{1}{8}\hat{\phi}} \hat{\gamma}_{11} \hat{\psi}_{11}, \quad (84)$$

set

$$a_1 = \frac{3}{4} \Rightarrow a_2 = 2^{-7} 3^{-1}, \quad (85)$$

and get the action [35]

$$\begin{aligned} \hat{S}^{IIA} = & \frac{1}{2} \int d^{10}x \sqrt{|\hat{g}|} \left\{ e^{-2\hat{\phi}} \left[-\hat{R} + 4 \left(\partial \hat{\phi} \right)^2 - \frac{3}{4} \hat{H}^2 \right] \right. \\ & \left. + \frac{1}{4} \hat{F}^2 + \frac{3}{4} \hat{G}^2 + \frac{1}{64} \frac{\hat{\epsilon}}{\sqrt{-\hat{g}}} \partial \hat{C} \partial \hat{C} \hat{B} \right\}, \end{aligned} \quad (86)$$

¹¹Observe that these definitions differ from those in Ref. [44] not only by powers of $e^{\hat{\phi}}$ but also, in the gravitino case, by the relative factor between $\hat{\psi}_{\hat{a}}$ and $\hat{\gamma}_{\hat{a}} \hat{\gamma}_{11} \hat{\psi}_{11}$. Both differences are caused by the fact that we are working in the string frame.

and supersymmetry transformation laws of the gravitino $\hat{\psi}_{\hat{a}}$ and dilatino $\hat{\lambda}$ fields

$$\delta_{\epsilon}\hat{\psi}_{\hat{a}} = \partial_{\hat{a}}\hat{\epsilon} - \frac{1}{4}\left(\hat{\omega}_{\hat{a}\hat{b}\hat{c}} - \frac{3}{2}\hat{H}_{\hat{a}\hat{b}\hat{c}}\hat{\gamma}_{11}\right)\hat{\gamma}^{\hat{b}\hat{c}}\hat{\epsilon} \quad (87)$$

$$- \frac{i}{16}e^{\hat{\phi}}\left(\hat{\gamma}_{\hat{a}}^{\hat{b}\hat{c}} - 2\delta_{\hat{a}}^{\hat{b}}\hat{\gamma}^{\hat{c}}\right)\hat{\gamma}_{11}\hat{F}_{\hat{b}\hat{c}}\hat{\epsilon} \quad (88)$$

$$- \frac{i}{32}e^{\hat{\phi}}\left(\hat{\gamma}_{\hat{a}}^{\hat{b}\hat{c}\hat{d}\hat{e}} - 4\delta_{\hat{a}}^{\hat{b}}\hat{\gamma}^{\hat{c}\hat{d}\hat{e}}\right)\hat{G}_{\hat{b}\hat{c}\hat{d}\hat{e}}\hat{\epsilon}, \quad (89)$$

$$\delta_{\epsilon}\hat{\lambda} = \left(\not{\partial}\hat{\phi} - \frac{1}{4}\hat{H}\hat{\gamma}_{11}\right)\hat{\epsilon} - \frac{3i}{8}e^{\hat{\phi}}\left(\hat{F}\hat{\gamma}_{11} + \frac{1}{6}\hat{G}\right)\hat{\epsilon}. \quad (90)$$

$$\delta_{\epsilon}\hat{\lambda} = \left(\not{\partial}\hat{\phi} - \frac{1}{4}\hat{H}\hat{\gamma}_{11}\right)\hat{\epsilon} - \frac{3i}{8}e^{\hat{\phi}}\left(\hat{F}\hat{\gamma}_{11} + \frac{1}{6}\hat{G}\right)\hat{\epsilon}. \quad (91)$$

B.2 Truncation to the $N = 1$ theory

When all RR fields ($\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}}, \hat{A}_{\hat{\mu}}$) are set to zero (which is a consistent truncation), the bosonic action of the type IIA theory Eq. (86) reduces to that of the $N = 1$ theory, which only contains the NS-NS fields $\hat{g}_{\hat{\mu}\hat{\nu}}, \hat{B}_{\hat{\mu}\hat{\nu}}, \hat{\phi}$:

$$\hat{S}^{N=1} = \frac{1}{2} \int d^{10}x \sqrt{|\hat{g}|} e^{-2\hat{\phi}} \left[-\hat{R} + 4 \left(\partial\hat{\phi} \right)^2 - \frac{3}{4}\hat{H}^2 \right] \quad (92)$$

In the supersymmetry transformation rules, though, both chiralities are still present. If we split all spinors in their positive $(+)$ and negative $(-)$ chirality halves

$$\hat{\epsilon} = \hat{\epsilon}^{(+)} + \hat{\epsilon}^{(-)}, \quad \hat{\gamma}_{11} \hat{\epsilon}^{(\pm)} = \pm \hat{\epsilon}^{(\pm)}, \quad (93)$$

etc. we get

$$\delta_{\epsilon}\hat{\psi}_{\hat{a}}^{(\pm)} = \hat{\nabla}_{\hat{a}}^{(\pm)}\hat{\epsilon}^{(\pm)}, \quad (94)$$

$$\delta_{\epsilon}\hat{\lambda}^{(\pm)} = \left(\not{\partial}\hat{\phi} \pm \frac{1}{4}\hat{H}\right)\hat{\epsilon}^{(\pm)}, \quad (94)$$

where $\hat{\nabla}_{\hat{a}}^{(\pm)}$ are the covariant derivatives associated to the two torsionful spin connections

$$\hat{\Omega}_{\hat{a}\hat{b}\hat{c}}^{(\pm)} = \hat{\omega}_{\hat{a}\hat{b}\hat{c}} \mp \frac{3}{2}\hat{H}_{\hat{a}\hat{b}\hat{c}}. \quad (95)$$

Eqs. (94) are just the supersymmetry transformation rules of the gravitino and dilatino of two $N = 1$ theories of different chiralities. Both are related by a

change of sign of the axion and a change in the chirality of the supersymmetry parameter $\hat{\epsilon}$. This transformation is a duality symmetry of the $N = 2A$ theory and a string/string duality symmetry between two different $N = 1$ theories of opposite chirality in its own right[15]: C duality.

B.3 Further reduction from $D = 10$ to $D = 4$

The dimensional reduction of $N = 1, D = 10$ supergravity to $10 - d$ dimensions in the string frame was performed in Ref. [38] (in the canonical frame it was done in Ref. [36]). Here we just quote their result for the four-dimensional action using our conventions, and give the relations between ten- and four-dimensional fields that allow us to uplift four-dimensional solutions to ten dimensions.

The four-dimensional action is

$$S = \frac{1}{2} \int d^4x \sqrt{|g|} e^{-2\phi} \left\{ -R + 4(\partial\phi)^2 - \frac{3}{4}H^2 \right. \\ \left. + \frac{1}{4} [\partial G_{mn} \partial G^{mn} - G^{mn} G^{pq} \partial B_{mp} \partial B_{nq}] \right. \\ \left. - \frac{1}{4} [G_{mn} F^{(1)m} F^{(1)n} + G^{mn} \mathcal{F}_m \mathcal{F}_n] \right\}, \quad (96)$$

where the vector and axion field-strengths are

$$F^{(1)m} = 2\partial A^{(1)m}, \quad H = \partial B - \frac{1}{2}A^{(1)m}F^{(2)}_m - \frac{1}{2}A^{(2)}_m F^{(1)m}, \\ F^{(2)}_m = 2\partial A^{(2)}_m, \quad \mathcal{F}_m = F^{(2)}_m + F^{(1)n}B_{nm}, \quad (97)$$

If we are given the four-dimensional fields $g_{\mu\nu}, B_{\mu\nu}, A^{(1)m}_{\mu}, A^{(2)}_{m\mu}, G_{mn}, B_{mn}$ and ϕ of a solution, the ten-dimensional fields of the corresponding ten-dimensional solutions can be found by using

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + A^{(1)m}_{\mu} A^{(1)n}_{\nu} G_{mn}, \quad \hat{g}_{mn} = G_{mn}, \\ \hat{B}_{\mu\nu} = B_{\mu\nu} + A^{(1)m}_{\mu} A^{(1)n}_{\nu} B_{mn} - A^{(1)m}_{\mu} A^{(2)}_{m\nu}, \quad \hat{B}_{mn} = B_{mn} \\ \hat{B}_{\mu m} = A^{(2)}_{m\mu} + A^{(1)n}_{\mu} B_{nm}, \quad \hat{g}_{\mu m} = A^{(1)n}_{\mu} G_{nm}, \\ \hat{\phi} = \phi + \frac{1}{4} \log \det |G|. \quad (98)$$

It is also useful to perform the dimensional reduction of the $N = 1, D = 10$ supersymmetry transformation rules to identify which are the six vector fields that belong to the $N = 4, D = 4$ gravity supermultiplet and the six vector fields that belong to the six additional $N = 4, D = 4$ vector supermultiplets. For this purpose, it is not necessary to reduce the spinor indices and thus we will keep the ten-dimensional gamma matrices and spinor indices. It is also sufficient to reduce the positive chirality theory (the rules of the theory with opposite ten-dimensional chirality can be obtained by a change of sign of $B_{\mu\nu}, B_{mn}$ and $A_{\mu m}^{(2)}$). The gravitini, dilatini and photini supersymmetry transformation rules are, respectively

$$\begin{aligned}\delta_\epsilon \hat{\psi}_a^{(+)} &= \nabla_a^{(+)} \hat{\epsilon}^{(+)} - \frac{1}{4} (\mathcal{F}_m^1 - \mathcal{F}_m) e_i^m \hat{\gamma}^i \hat{\epsilon}^{(+)} \\ &\quad + (\partial_a e_{mj} + \partial_a B_{mn} e_j^n) e_i^m \hat{\gamma}^{ij} \hat{\epsilon}^{(+)} , \\ \delta_\epsilon \left(\hat{\lambda}^{(+)} - \hat{\gamma}^i \hat{\psi}_i^{(+)} \right) &= \left\{ \partial \phi + \frac{1}{4} \mathcal{H} - \frac{1}{8} (\mathcal{F}_m^{(1)} - \mathcal{F}_m) e_i^m \hat{\gamma}^i \right\} \hat{\epsilon}^{(+)} , \quad (99) \\ \delta_\epsilon \hat{\psi}_i^{(+)} &= \frac{1}{8} (\mathcal{F}_m^{(1)} + \mathcal{F}_m) e_i^m \hat{\gamma}^i \hat{\epsilon}^{(+)} \\ &\quad - \frac{1}{4} (\partial G_{mn} + \partial B_{mn}) e_i^m e_j^n \hat{\gamma}^j \hat{\epsilon}^{(+)} . \quad (100)\end{aligned}$$

This means that the supergravity vector fields are in the combinations

$$F^{(1)m} - \mathcal{F}_m , \quad (101)$$

and the matter vector fields are in the combinations

$$F^{(1)m} + \mathcal{F}_m . \quad (102)$$

These combinations interchange their roles under C duality.

C Spin connection coefficients

Most of the ten-dimensional metrics we have met are of the general form

$$\begin{aligned}ds^2 &= V^d dt^2 - W^b d\vec{x}^2 - \left(V^{-\frac{d}{2}} dx^4 + n V^{\frac{d}{2}} dt \right)^2 \\ &\quad - W^c \left(dx^5 + m W_{\underline{i}} dx^{\underline{i}} \right)^2 - dx^I dx^I , \quad (103)\end{aligned}$$

with $I = 6, \dots, 9$. We ignore these directions since they are flat. Choosing the zehnbein one-form basis

$$\begin{aligned}\hat{e}^0 &= V^{\frac{d}{2}} dt , & \hat{e}^4 &= V^{-\frac{d}{2}} dx^4 + n V^{\frac{d}{2}} dt , \\ \hat{e}^i &= W^{\frac{b}{2}} dx^{\underline{i}} , & \hat{e}^5 &= W^{\frac{c}{2}} (dx^5 + m W_{\underline{i}} dx^{\underline{i}}) ,\end{aligned} \quad (104)$$

we get the following non-vanishing components of the spin connection one-form

$$\hat{\omega}^{0i} = \frac{d}{2} W^{-\frac{b}{2}} V^{-1} \partial_{\underline{i}} V (\hat{e}^0 - n \hat{e}^4) , \quad (105)$$

$$\hat{\omega}^{04} = -\frac{nd}{2} W^{-\frac{b}{2}} V^{-1} \partial_{\underline{i}} V \hat{e}^i , \quad (106)$$

$$\hat{\omega}^{ij} = +b W^{-\frac{b}{2}-1} \delta_{k[i} \partial_{\underline{j}} W \hat{e}^k - m W^{\frac{c}{2}-b} \partial_{[\underline{i}} W_{\underline{j}]} \hat{e}^5 , \quad (107)$$

$$\hat{\omega}^{i4} = -\frac{nd}{2} W^{-\frac{b}{2}} V^{-1} \partial_{\underline{i}} V (\hat{e}^0 - n \hat{e}^4) , \quad (108)$$

$$\hat{\omega}^{i5} = -m W^{\frac{c}{2}-b} \partial_{[\underline{i}} W_{\underline{j}]} \hat{e}^j - \frac{c}{2} W^{-\frac{b}{2}-1} \partial_{\underline{i}} W \hat{e}^5 . \quad (109)$$

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